# Fast Polynomial Approximations to Sine and Cosine

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**Abstract**

In this paper, polynomial approximations to sine and cosine are given which minimize the $L^2$ norm of the difference between polynomial approximations and either sine or cosine on certain intervals. These polynomial approximations are less accurate than the approximations from the C and Java libraries, but in some cases, they are much faster. Speed tests are shown to confirm this and an approximation of the accuracy of each polynomial approximation is given.

## 1 Introduction

When one is coding in the C or Java languages, it is common to use the language's standard math library for calculating sine and cosine. However, for certain applications, the standard library may not be sufficient. For embedded applications, there may not be much memory to store the program. Including the standard math library could make the program too large to store in memory. Second, if speed is an issue, it may be faster to use a polynomial approximation for sine and/or cosine. Even if the processor implements the sine and cosine functions in hardware, the hardware's implementation has to be accurate for a large range of problems. For instance the Intel®IA-64 architecture reduces the argument of sine/cosine to some small interval, and ensures the result is accurate to within 1 ULPS (units of last place) [4].

If such accuracy implementations are not required, using polynomials may be a better choice for approximating sine and cosine. The highest degree polynomial given in this paper is a degree 21 polynomial of which only 11 coefficients are necessary to evaluate the polynomial. Thus, the memory requirements necessary to implement such polynomials are quite small, as opposed to either importing the standard math library or using methods requiring tables [6]. Also, polynomials only require multiplication, addition, and subtraction of floating point numbers, which normally take very few CPU cycles for processors with floating point cores.

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## 2 Theory of Polynomial Approximation using the $L^2$ Norm

The theory of using $L^2$ polynomial approximations for functions is not new [1], but for completeness, the theory is quickly presented. Let $P_n$ denote the set of polynomials of degree at most $n$. The $L^2$ inner product between two functions $f$ and $g$ defined from $[a, b]$ to $\mathbf{R}$ is defined by:

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx. \tag{1}$$

The $L^2$ inner product induces the $L^2$ norm which is given by:

$$\|f\| = \langle f, f\rangle^{1/2}. \tag{2}$$

To approximate a function $f : [a, b] \to \mathbf{R}$ by polynomials in $P_n$, one can find:

$$\min_{p\in P_n} \|f - p\|^2 = \min_{p\in P_n} \int_a^b \left(f(x) - p(x)\right)^2 dx. \tag{3}$$

To find $p \in P_n$ which solves the minimization problem (3), first construct an orthonormal basis of $P_n$. To find an orthonormal basis, start with the set of polynomials $\{1, x, x^2, ..., x^n\}$, which forms a basis $P_n$. To make this into an orthonormal basis $\{p_0(x), p_1(x), ..., p_n(x)\}$, one can use the Gram-Schmidt procedure. The procedure is given by the following recursion:

$$\begin{aligned} p_0(x) &= 1/\|1\| \\ \widehat{p}_k(x) &= x^k - \sum_{i=0}^{k-1} \langle x^k, p_i(x)\rangle p_i(x) \\ p_k(x) &= \widehat{p}_k(x)/\|\widehat{p}_k(x)\|. \end{aligned}$$

Once an orthonormal basis for $P_n$ is created, then the algorithm:

$$p = \langle f, p_0\rangle p_0 + ... + \langle f, p_n\rangle p_n \tag{4}$$

gives the polynomial $p$ of degree n which minimizes $\|f - p\|$. Replacing $f$ with sine or cosine, gives the best $L^2$ approximation by an $n$ degree polynomial for sine or cosine respectively.

## 3 Polynomial Approximations

The code used to find the polynomial approximations was written in C++ and compiled using GCC version 4.3.2. Also the GMP library [3] was used to implement higher precision arithmetic than the native CPU allowed. This was essential in obtaining approximations close to IEEE double floating point precision. Appendix A contains the coefficients of the approximations for sine and cosine on different intervals as well as a pointwise error estimate for each polynomial. The error approximation is calculated as the maximum error between the polynomial and the C math library function, sine or cosine, at 10,000 equally spaced points of the interval of approximation. This algorithm is shown below.

```
Let p be the approximating polynomial.
Let f be the C math library function (either sine or cosine).
Let low and high be the interval bounds for the approximation.
Let error be the pointwise error returned.

error = 0;
h = (high - low) / 10000;

for i = 0; i <= 10000; i = i + 1
    x = low + i * h;
    tempError = abs(f(x) - p(x));
    if(tempError > error)
        error = tempError;
end for loop
```

# 4 Speed Tests

Some speed tests were done to test the C and Java standard math libraries against the 21 and 11 degree polynomial approximations for sine given in Section A.1. The version of the Java compiler/runtime environment was 1.6.0_19, and the version of GCC used to compile the speed tests was 4.3.2. The processor of the machine was an Intel Xeon 3.2 GHz processor. The algorithm used to test the speed of the math libraries versus the polynomial approximation, calculates various values of sine 100,000,000 times.

It should be noted that to get the fastest speed and greatest numerical accuracy, one should use Horner's scheme of evaluating polynomials [2]. Below is an example of using this algorithm for computing the 11 degree polynomial approximation of sine on $[-\pi, \pi]$.

```
// Compute sin(x)
double x2 = x * x;
return
(((((-2.0534285628974660072 7e-08*x2 + 2.7040521830779904008 4e-06)*x2
     - 1.9812576341780668190 9e-04)*x2 + 8.3325581475518801046 4e-03)*x2
     - 1.6666577219696162398 3e-01)*x2 + 9.9999970704415654668 5e-01)*x;
```

| Method | Time (ms) | Speedup | Accuracy |
|---|---|---|---|
| C math library | 7520 | | |
| C 21 deg polynomial | 3930 | 1.91x | 7.532e-16 |
| C 11 deg polynomial | 2100 | 3.58x | 3.056e-07 |
| Java math library | 19487 | | |
| Java 21 deg polynomial | 5201 | 3.75x | 7.532e-16 |
| Java 11 deg polynomial | 3609 | 5.40x | 3.056e-07 |

Even with the CPU implementing the sine function in hardware (which is how the C standard math library computes sine), using a polynomial approximation is still faster. When it comes to Java performance, the polynomial approximations are much faster than using the Java math library. This seems to be from the Java math library not solely using the CPU's sine function, but rather using its own algorithm to ensure a higher precision than is guaranteed by x86 processors [5]. Hence, for Java, polynomial approximations may be very useful for applications needing very fast performance.

# 5 Conclusion

To conclude, I would like to say that using the standard math library in C and Java is the standard practice and should not be changed unless circumstances dictate the necessity. Only in situations requiring a very small memory footprint or needing an extra boost of speed, should these approximations be used. These approximations do not take into account reducing the input to sine and cosine to a small interval, nor do the approximations account for obtaining accuracy to within 1 ULPS. However, with all of that being said, the approximations given may be useful in several areas of computing where either memory or speed performance is critical.

# A Equations

## A.1 Sine on $[-\pi, \pi]$

```
 + 3.0396355092701331 4332e-01x
Pointwise Error Estimate: 9.549296585513718 92153e-01

 - 9.338769728318536 19134e-02x^3 + 8.569833277952495 06462e-01x
Pointwise Error Estimate: 2.033122536480397 71447e-01

 + 5.643117976346810 35370e-03x^5 - 1.552714106334286 44799e-01x^3
 + 9.878621355746738 06965e-01x
Pointwise Error Estimate: 1.597729356818959 54411e-02

 - 1.477408807973185 21837e-04x^7 + 7.998581437431325 51201e-03x^5
 - 1.658384526980308 73892e-01x^3 + 9.994501938932620 89505e-01x
Pointwise Error Estimate: 6.649733497747657 86287e-04

 + 2.173262174985967 29611e-06x^9 - 1.931627964073568 30500e-04x^7
 + 8.312388874178845 98346e-03x^5 - 1.666325950720867 45320e-01x^3
 + 9.999845941934943 65437e-01x
Pointwise Error Estimate: 1.723335286839274 37958e-05

 - 2.053428562897466 00727e-08x^11 + 2.704052183077990 40084e-06x^9
```

```
 - 1.9812576341780668190 9e-04x^7 + 8.3325581475518801046 4e-03x^5
 - 1.6666577219696162398 3e-01x^3 + 9.9999970704415654668 5e-01x
Pointwise Error Estimate: 3.0557307311910208451 0e-07

 + 1.3533382554521859927 2e-10x^13 - 2.4701642548052786903 2e-08x^11
 + 2.7532295533044991116 3e-06x^9 - 1.9840311266901899669 0e-04x^7
 + 8.3333145143308074975 5e-03x^5 - 1.6666665043706634628 6e-01x^3
 + 9.9999995973569972699e-01x
Pointwise Error Estimate: 3.9451710261481244517 1e-09

 - 6.5807548917512165702 6e-13x^15 + 1.5885000479150482342 3e-10x^13
 - 2.5036891439210308312 0e-08x^11 + 2.7556559875210270400 8e-06x^9
 - 1.9841248360434080585 9e-04x^7 + 8.3333301181570639096 4e-03x^5
 - 1.6666666645135216797 4e-01x^3 + 9.9999999958141380079 e-01x
Pointwise Error Estimate: 3.8769880570244730401 6e-11

 + 2.4592852429015300225 9e-15x^17 - 7.5810626051136155481 1e-13x^15
 + 1.6052198438515305917 2e-10x^13 - 2.5051686135970637821 0e-08x^11
 + 2.7557303484398611128 0e-06x^9 - 1.9841269497124211824 1e-04x^7
 + 8.3333332926687803703e-03x^5 - 1.6666666664489411560e-01x^3
 + 9.9999999999659411867e-01x
Pointwise Error Estimate: 2.9973088999418044269 3e-13

 - 7.2863896593544838237 5e-18x^19 + 2.7916435400997537456 6e-15x^17
 - 7.6447930778567702375 9e-13x^15 + 1.6058869592896627810 5e-10x^13
 - 2.5052100301218831635 3e-08x^11 + 2.7557318989267188436 5e-06x^9
 - 1.9841269837184033492 9e-04x^7 + 8.3333333329438515047 e-03x^5
 - 1.6666666666649732329e-01x^3 + 9.9999999999997848557e-01x
Pointwise Error Estimate: 2.0939634747550674400 8e-15

 + 2.4785230623349397411 5e-20x^21 - 8.5393228791656423823 1e-18x^19
 + 2.8187535034686122663 3e-15x^17 - 7.6480713449381593227 5e-13x^15
 + 1.6059112256720897789 5e-10x^13 - 2.5052111623008981391 3e-08x^11
 + 2.7557319319685576035 9e-06x^9 - 1.9841269842967257032 0e-04x^7
 + 8.3333333334987771150e-03x^5 - 1.6666666666674074058e-01x^3
 + 1.0000000000000098216e+00x
Pointwise Error Estimate: 7.5317122452918980391 1e-16
```

## A.2 Cosine on $[-\pi, \pi]$

```
 - 2.3098460073039754175 6e-01x^2 + 7.5990887731753328582 9e-01
Pointwise Error Estimate: 5.1981775463506657165 8e-01

 + 2.6159820382171035154 9e-02x^4 - 4.5228781076661098968 6e-01x^2
 + 9.7832639089239478225 5e-01
```

```
Pointwise Error Estimate: 6.262894823123075214 39e-02

 - 9.928632951930131735 83e-04x^6 + 3.952232212933064313 94e-02x^4
 - 4.962486794510545599 90e-01x^2 + 9.989871710373326691 23e-01
Pointwise Error Estimate: 3.485548232570485831 21e-03

 + 1.906526688400742463 05e-05x^8 - 1.344107693492853217 33e-03x^6
 + 4.152230862509107675 16e-02x^4 - 4.998376022729957344 37e-01x^2
 + 9.999710946061826873 41e-01
Pointwise Error Estimate: 1.128008192238665084 04e-04

 - 2.219417827863537270 22e-07x^10 + 2.425324013810330274 81e-05x^8
 - 1.386275070625736737 56e-03x^6 + 4.166103373540211074 29e-02x^4
 - 4.999955824990650484 20e-01x^2 + 9.999994437395372108 53e-01
Pointwise Error Estimate: 2.395659169117912526 21e-06

 + 1.736914894508212936 70e-09x^12 - 2.711337719408011385 03e-07x^10
 + 2.477342457309302502 60e-05x^8 - 1.388797042704520541 54e-03x^6
 + 4.166652436776862304 61e-02x^4 - 4.999999177286145919 00e-01x^2
 + 9.999999229082749171 1e-01
Pointwise Error Estimate: 3.601408303620010534 57e-08

 - 9.775071315270064981 14e-12x^14 + 2.062075039158135195 67e-09x^12
 - 2.753699185737995458 60e-07x^10 + 2.480069137186652602 56e-05x^8
 - 1.388886746876913397 50e-03x^6 + 4.166666415903619851 36e-02x^4
 - 4.999999988865269270 02e-01x^2 + 9.999999991936547995 7e-01
Pointwise Error Estimate: 4.037393412416881857 87e-10

 + 4.148697218699475724 36e-14x^16 - 1.136007777959586757 06e-11x^14
 + 2.086618973582619036 87e-09x^12 - 2.755672984371603830 39e-07x^10
 + 2.480156799939217515 41e-05x^8 - 1.388888853442763718 09e-03x^6
 + 4.166666634151863687 3e-02x^4 - 4.999999999885605719 10e-01x^2
 + 9.999999999934074548 5e-01
Pointwise Error Estimate: 3.506890449503329007 91e-12

 - 1.375758388868985652 59e-16x^18 + 4.742258141855808015 53e-14x^16
 - 1.146659071597660345 38e-11x^14 + 2.087647606805018461 30e-09x^12
 - 2.755730746905812418 11e-07x^10 + 2.480158700250424350 69e-05x^8
 - 1.388888888452694120 33e-03x^6 + 4.166666666344487602 0e-02x^4
 - 4.999999999990801701 2e-01x^2 + 9.999999999999568580 2e-01
Pointwise Error Estimate: 2.434511553402737398 78e-14

 + 3.683962162224004778 86e-19x^20 - 1.552893183778014966 07e-16x^18
 + 4.778404397145566115 32e-14x^16 - 1.147066784990298602 38e-11x^14
 + 2.087675347807698715 95e-09x^12 - 2.755731912732797484 39e-07x^10
```

```
 + 2.4801587300079678004 8e-05x^8 - 1.3888888888779804960e-03x^6
 + 4.1666666666665603386e-02x^4 - 5.0000000000000154115e-01x^2
 + 1.0000000000000001607e+00
Pointwise Error Estimate: 4.31834595667259421673e-16
```

## A.3 Sine on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

```
 + 7.7403682639678774042 6e-01x
Pointwise Error Estimate: 2.15854203708053257327e-01

 - 1.4506181330686809332 1e-01x^3 + 9.8879223305330797352 4e-01x
Pointwise Error Estimate: 9.03713224888425787837e-03

 + 7.5742472935399066958 2e-03x^5 - 1.6582704231332937839 8e-01x^3
 + 9.9977140781715433292 6e-01x
Pointwise Error Estimate: 1.60759692603255652753e-04

 - 1.8447220592329031905 0e-04x^7 + 8.3095169397830324691 4e-03x^5
 - 1.6665168101066444316 2e-01x^3 + 9.9999748719816896489 1e-01x
Pointwise Error Estimate: 1.58300968787476130146e-06

 + 2.5981104440855580966 6e-06x^9 - 1.9804755300924134690 6e-04x^7
 + 8.3329640182185505958 6e-03x^5 - 1.6666651520223623015 6e-01x^3
 + 9.9999982782301183838 e-01x
Pointwise Error Estimate: 9.90130157423991881556e-09

 - 2.3843684079972302917 6e-08x^11 + 2.7521940763875024650 2e-06x^9
 - 1.9840772933712366070 0e-04x^7 + 8.3333295329348954259 e-03x^5
 - 1.6666666568700398558 8e-01x^3 + 9.9999999919514177715 e-01x
Pointwise Error Estimate: 4.28286319713321221399e-11

 + 1.5389296582941170412 3e-10x^13 - 2.5028396980392793732 4e-08x^11
 + 2.7556891612846819267 4e-06x^9 - 1.9841265720930700175 6e-04x^7
 + 8.3333331302746573676 e-03x^5 - 1.6666666662275904351 e-01x^3
 + 9.9999999999727077895 e-01x
Pointwise Error Estimate: 1.35764985157529617585e-13

 - 7.3664454192453278740 3e-13x^15 + 1.6047392248768257333 1e-10x^13
 - 2.5051851666625091008 7e-08x^11 + 2.7557316008383331990 9e-06x^9
 - 1.9841269818489884322 5e-04x^7 + 8.3333333324810981519 e-03x^5
 - 1.6666666666652434127 e-01x^3 + 9.9999999999999312880 e-01x
Pointwise Error Estimate: 3.625226103071141745 20e-16

 + 2.7275344797416021612 1e-15x^17 - 7.6437997634064069223 4e-13x^15
 + 1.6058981952572145997 0e-10x^13 - 2.5052108048572423161 5e-08x^11
```

```
 + 2.7557319229942639 1309e-06x^9 - 1.98412698413826558397e-04x^7
 + 8.33333333333406623879e-03x^5 - 1.66666666666666861949e-01x^3
 + 1.00000000000000001550e+00x
Pointwise Error Estimate: 1.61698584449416341000e-16
```

## A.4 Cosine on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

```
 + 6.36619772367581343076e-01
Pointwise Error Estimate: 6.36619772367581281845e-01

 - 4.17697757006468189227e-01x^2 + 9.80162407440597336452e-01
Pointwise Error Estimate: 5.04654977784507049078e-02

 + 3.72093273724652635657e-02x^4 - 4.963923301200805975 72e-01x^2
 + 9.99579515069196269871e-01
Pointwise Error Estimate: 1.31345589781311754367e-03

 - 1.27871387439538728389e-03x^6 + 4.151173649148801572 26e-02x^4
 - 4.99930919784780092977e-01x^2 + 9.99995282689772072939e-01
Pointwise Error Estimate: 1.70004880294522700790e-05

 + 2.32376487283845000877e-05x^8 - 1.385742194470348976 48e-03x^6
 + 4.16640913730533583858e-02x^4 - 4.999992689852175817 92e-01x^2
 + 9.99999672701154672 33e-01
Pointwise Error Estimate: 1.31691512104005950656e-07

 - 2.61150902081161051732e-07x^10 + 2.476377404632911558 77e-05x^8
 - 1.38884324659041369860e-03x^6 + 4.166664188619104157 58e-02x^4
 - 4.99999951166316793 18e-01x^2 + 9.99999998457054688 07e-01
Pointwise Error Estimate: 6.79139330959177216253e-10

 + 1.99428671619384460126e-09x^12 - 2.752711866765731274 33e-07x^10
 + 2.48011030524864844432e-05x^8 - 1.388888491461715687 22e-03x^6
 + 4.16666651205396044 43e-02x^4 - 4.999999999775821802 12e-01x^2
 + 9.99999999994736048 87e-01
Pointwise Error Estimate: 2.49950603831395964289e-12

 - 1.10226779988697263833e-11x^14 + 2.085951918854542170 39e-09x^12
 - 2.75569737441483945459e-07x^10 + 2.480158347280332996 00e-05x^8
 - 1.38888888659158848090e-03x^6 + 4.166666665992468268 2e-02x^4
 - 4.99999999999250063 05e-01x^2 + 9.99999999999986402 56e-01
Pointwise Error Estimate: 6.95651296844558053755e-15

 + 4.61234214964119671037e-14x^16 - 1.146321340889423420 94e-11x^14
 + 2.08765734915654935676e-09x^12 - 2.755731661664192142 40e-07x^10
```

```
 + 2.4801587279821183 6808e-05x^8 - 1.3888888887868692476e-03x^6
 + 4.166666666642765080e-02x^4 - 4.999999999999786172e-01x^2
 + 9.99999999999999 6961e-01
Pointwise Error Estimate: 2.3896034651030097 9755e-16

 - 9.026635538224424 31771e-17x^18 + 4.709704055183378853 40e-14x^16
 - 1.146758124313848 08730e-11x^14 + 2.087667894588372528 93e-09x^12
 - 2.755731809707947 53551e-07x^10 + 2.480158729199729530 19e-05x^8
 - 1.388888888884295 01741e-03x^6 + 4.166666666666556 57082e-02x^4
 - 4.999999999999989 9778e-01x^2 + 9.999999999999999 8600e-01
Pointwise Error Estimate: 2.375446113001997 32860e-16
```

## A.5 Sine on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

```
 + 9.396585011617269 15864e-01x
Pointwise Error Estimate: 3.089927984667190 68148e-02

 - 1.610345324948069 62322e-01x^3 + 9.992590185606561 95678e-01x
Pointwise Error Estimate: 3.074780884325728 02901e-04

 + 8.137645277508843 77365e-03x^5 - 1.666119866368489 98403e-01x^3
 + 9.999962587295915 36999e-01x
Pointwise Error Estimate: 1.336351388569383 10381e-06

 - 1.948420397482192 88187e-04x^7 + 8.331795714592215 45387e-03x^5
 - 1.666664237961760 28737e-01x^3 + 9.999999879366984 85360e-01x
Pointwise Error Estimate: 3.246725544479201 24733e-09

 + 2.715528987697245 20716e-06x^9 - 1.983892569781363 79687e-04x^7
 + 8.333273859390158 8006e-03x^5 - 1.666666605563541 0211e-01x^3
 + 9.999999998261172 2898e-01x
Pointwise Error Estimate: 5.033687556212780 66309e-12

 - 2.474479958457468 76115e-08x^11 + 2.755505702157422 12602e-06x^9
 - 1.984126187492731 90525e-04x^7 + 8.333333197628195 7031e-03x^5
 - 1.666666666568221 7796e-01x^3 + 9.999999999997976 3254e-01x
Pointwise Error Estimate: 5.348102508938489 62010e-15

 + 1.588936655631989 35090e-10x^13 - 2.505060202065976 30293e-08x^11
 + 2.755731243188397 48870e-06x^9 - 1.984126982493000 45644e-04x^7
 + 8.333333333313239 07170e-03x^5 - 1.666666666666655 80149e-01x^3
 + 9.999999999999999 82832e-01x
Pointwise Error Estimate: 1.460728296373019 56081e-16

 - 8.047544901538365 39537e-13x^15 + 1.606910230804160 13690e-10x^13
```

```
 - 2.5052203476906868749e-08x^11 + 2.7557319676181301871e-06x^9
 - 1.9841269842416013745e-04x^7 + 8.3333333333481157087e-03x^5
 - 1.6666666666666747430e-01x^3 + 1.0000000000000000098e+00x
Pointwise Error Estimate: 1.4588609689846395765e-16
```

## A.6 Cosine on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

```
 + 9.0031631615710606955e-01
Pointwise Error Estimate: 1.9320953497055849681e-01

 - 4.7834361020928192648e-01x^2 + 9.9867177866865818532e-01
Pointwise Error Estimate: 3.5013900525457347249e-03

 + 4.0512153766521833025e-02x^4 - 4.9976355294740725076e-01x^2
 + 9.9999306842565370875e-01
Pointwise Error Estimate: 2.2049299442327732795e-05

 - 1.3605886670755467091e-03x^6 + 4.1656625803167383019e-02x^4
 - 4.9999887557761143038e-01x^2 + 9.9999980750852337007e-01
Pointwise Error Estimate: 7.0141193282694304127e-08

 + 2.4401928879421992057e-05x^8 - 1.3886863619529448706e-03x^6
 + 4.1666625074790788121e-02x^4 - 4.9999997041875317281e-01x^2
 + 9.9999999966839549719e-01
Pointwise Error Estimate: 1.3441709522668551422e-10

 - 2.7190052216493075315e-07x^10 + 2.4799164986526555669e-05x^8
 - 1.3888881556486555739e-03x^6 + 4.1666665669563568388e-02x^4
 - 4.9999999995081518538e-01x^2 + 9.9999999999961114403e-01
Pointwise Error Estimate: 1.7211999413013462766e-13

 + 2.0639522959913836152e-09x^12 - 2.7555390780578459562e-07x^10
 + 2.4801579549316724005e-05x^8 - 1.3888888872953958540e-03x^6
 + 4.1666666665114655876e-02x^4 - 4.9999999999994366236e-01x^2
 + 9.9999999999999966857e-01
Pointwise Error Estimate: 1.7706142843396318676e-16

 - 1.1338834694986236418e-11x^14 + 2.0875258908192056036e-09x^12
 - 2.7557310242534375239e-07x^10 + 2.4801587271190433731e-05x^8
 - 1.3888888888314249467e-03x^6 + 4.1666666666661080011e-02x^4
 - 4.9999999999999977484e-01x^2 + 9.9999999999999999822e-01
Pointwise Error Estimate: 1.3002042434958564366e-16
```

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